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Hermite–Lagrange Interpolation and Schur's Expansion of sin πx

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With the help of Taylor's formula in the theory of right-invertible operators, a new constructive approach to the Hermite–Lagrange 2-point interpolation polynomial with explicit remainder and to Schur's expansion of sin πx is presented. This improves earlier results of I. Schur, L. Carlitz, G. C. Rota, D. Kahaner, A. Odlyzko, S. Wrigge, and A. Fransén. Also, we prove: If $f \in C^\infty[0, 1]$ is completely convex, then

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(a_n + b_n x)(x(1-x))^n, \qquad a_n \geqslant 0,\ a_n + b_n \geqslant 0,$$

uniformly in [0, 1].

1. INTRODUCTION

It was proved by I. Schur (see [4, pp. 128 and 307]) that in the convergent expansion

$$\sin \pi x = \sum_{n=1}^{\infty} \frac{1}{n!} a_n (x(1-x))^n, \qquad (x \in [0, 1]), \tag{1}$$

the coefficients a_n are positive, but no explicit expression for a_n was given. Later, L. Carlitz [3] found the explicit formula

$$a_{r+1} = \sum_{s=0}^{[r/2]} (-1)^s \frac{(2r-2s)!}{(r-2s)!(2s)!} \pi^{2s+1} \qquad (r = 0, 1, \ldots), \tag{2}$$

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but it is not clear from his result that all coefficients are positive. Now G. C. Rota, D. Kahaner, and A. Odlyzko [5] have shown that

$$a_n = \frac{\pi}{(n-1)!} \int_0^{\pi/2} (t(\pi - t))^{n-1} \sin t \, dt. \tag{3}$$

From this, the positivity of a_n can be inferred. But the expansion of $\exp ax$ in powers of $x(1-x)$, given in [5], is not correct. Also, S. Wrigge and A. Fransén [8] have applied partial sums of (1) for the rapid calculation of $\sin \pi x$ and other trigonometric functions. This was done independently of the earlier results [3–5].

In the following note, a new constructive approach to such expansions is presented by the help of Taylor's formula in the theory of right-invertible operators [6]. Note that there is a close connection between Hermite–Lagrange 2-point interpolation and Schur's expansion. Further, we prove that for every completely convex function $f \in C^\infty[0, 1]$, the series with non-negative terms

$$(1-x)f(0) + xf(1) + \sum_{n=1}^{\infty} \frac{1}{n!} (a_n + b_n x)(x(1-x))^n \tag{4}$$

converges uniformly in $[0, 1]$ to $f(x)$, where the constants a_n and b_n are defined by (15).

2. Taylor's Formula

Let X be a Banach space. We consider only linear operators defined on linear subsets of X and with ranges in X. The domain, image, and kernel of an operator A will be denoted by $\operatorname{dom} A$, $\operatorname{im} A$, and $\ker A$, respectively. By I we denote the identity. In the following let $D\colon \operatorname{dom} D \to X$ ($\operatorname{dom} D \subseteqq X$) be a right-invertible operator.

Remark. Assume that $T_n\colon X \to \operatorname{dom} D^n$ ($n = 1, 2, \ldots$) is a bounded right inverse of D^n with the corresponding projector $P_n = I - T_n D^n\colon \operatorname{dom} D^n \to \ker D^n$. Then we obtain the following relation on $\operatorname{dom} D$:

$$T_n - T_{n+1} D = (T_n D^n - T_{n+1} D^{n+1})\, T_n = (P_{n+1} - P_n)\, T_n = P_{n+1} T_n,$$

where $Q_n = P_{n+1} T_n$ maps $\operatorname{dom} D$ into $\ker D^{n+1}$. The following result is the converse statement. This will be a useful method of construction of right inverses.

THEOREM 1. *Let* $T_n\colon X \to \operatorname{dom} D^n$ ($n = 1, 2, \ldots$) *be bounded operators and let* $P_n\colon \operatorname{dom} D^n \to \ker D^n$ ($n = 1, 2, \ldots$) *be projectors onto* $\ker D^n$. *If*

(i) $DT_1 = I$,

(ii) $T_{n+1}D = T_n - Q_n$ $(n = 1, 2, ...)$ *on* $\operatorname{dom} D$ *with an operator* Q_n: $\operatorname{dom} D \to \ker D^{n+1}$,

(iii) $P_n T_n = 0$ $(n = 1, 2, ...)$,

then T_n *is a right inverse of* D^n *with the property*

$$P_n = I - T_n D^n. \tag{5}$$

Further, it holds

$$\begin{aligned} Q_n &= P_{n+1} T_n && \textit{on } \operatorname{dom} D, \\ P_{n+1} - P_n &= Q_n D^n && \textit{on } \operatorname{dom} D^{n+1}. \end{aligned} \tag{6}$$

Proof. By (i) and (iii), T_1 is a right inverse of D with $P_1 = I - T_1 D$. Assume that T_n is a right inverse of D^n with (5). Note that a right inverse T_n of D^n is uniquely determined by P_n. Then by (i) and (ii),

$$D^{n+1} T_{n+1} = D^{n+1}(T_n T_1 - Q_n T_1) = D(D^n T_n)\, T_1 = I \tag{7}$$

on X. Since T_n is a right inverse of D^n, it follows that T_n maps $\operatorname{dom} D$ into $\operatorname{dom} D^{n+1}$. By (ii) and (iii) we obtain that on $\operatorname{dom} D$,

$$0 = P_{n+1} T_{n+1} D = P_{n+1} T_n - Q_n,$$

i.e., $Q_n = P_{n+1} T_n$ on $\operatorname{dom} D$. By (iii) and (7), we have $P_{n+1} = I - T_{n+1} D^{n+1}$. Hence for $n = 1, 2, ...$,

$$P_{n+1} - P_n = T_n D^n - T_{n+1} D^{n+1} = (T_n - T_{n+1} D)\, D^n = Q_n D^n.$$

This completes the proof. ∎

Let $f \in \operatorname{dom} D^N$ $(N \geqslant 2)$ be given. Then under the assumptions of Theorem 1, we obtain by (5) and (6) Taylor's formula (see [6])

$$f = P_N f + T_N D^N f$$

with

$$P_N f = P_1 f + \sum_{n=1}^{N-1} (P_{n+1} - P_n) f = P_1 f + \sum_{n=1}^{N-1} Q_n D^n f. \tag{8}$$

3. Hermite–Lagrange Interpolation

Now let $X = C[0, 1]$, $D = d^2/dt^2$, and $\operatorname{dom} D = C^2[0, 1]$. Further, let P_n: $C^{2n}[0, 1] \to \ker D^n$ be the following projector onto $\ker D^n$ mapping

$f \in C^{2n}[0, 1]$ to the corresponding Hermite–Lagrange 2-point interpolation polynomial $p = P_n f$ of order $2n-1$ with

$$p^{(j)}(0) = f^{(j)}(0), \qquad p^{(j)}(1) = f^{(j)}(1) \qquad (j = 0, ..., n-1).$$

In particular,

$$(P_1 f)(x) = (1-x) f(0) + x f(1) \qquad (x \in [0, 1]).$$

THEOREM 2. *The corresponding right inverse T_n of D^n $(n = 1, 2, ...)$ with $P_n = I - T_n D^n$ can be expressed in the form*

$$(T_n f)(x) = -((n-1)!)^{-2} \int_x^1 \int_0^1 f(st)(x(x-s)\, t(1-t))^{n-1} xt\, dt\, ds. \tag{9}$$

Further, the operator $Q_n = T_n - T_{n+1} D$: $\operatorname{dom} D \to \ker D^{n+1}$ is given by

$$(Q_n f)(x) = (n!(n-1)!)^{-1} (x(x-1))^n$$
$$\times \int_0^1 f(t)(t(1-t))^{n-1}(1-t-x+2xt)\, dt. \tag{10}$$

Remark. For given $f \in C[0, 1]$, $T_n f = y \in C^{2n}[0, 1]$ $(n = 1, 2, ...)$ is the unique solution of the boundary value problem

$$y^{(2n)}(x) = f(x), \qquad y^{(j)}(0) = y^{(j)}(1) = 0 \qquad (j = 0, ..., n-1).$$

Proof. We apply Theorem 1. By

$$\int_0^x (x-u) f(u)\, du = \int_0^x \int_0^1 f(st)\, xt\, dt\, ds$$

it follows that

$$(T_1 f)(x) = \int_0^x (x-u) f(u)\, du - x \int_0^1 (1-u) f(u)\, du$$
$$= -\int_x^1 \int_0^1 f(st)\, xt\, dt\, ds.$$

For the proof of the equation

$$T_n f = T_{n+1} Df + Q_n f \qquad (n = 1, 2, ...) \tag{11}$$

for $f \in C^2[0, 1]$, we replace t in the kernel of $T_n f$ by $-(t-1) + (d/dt)(t(t-1))$. Integrating the first term of $T_n f$ by parts with respect to s

and integrating the second term of $T_n f$ by parts with respect to t, we obtain

$$(T_n f)(x) = (n!(n-1)!)^{-1}(x(x-1))^n \int_0^1 f(t)(t(1-t))^{n-1}(1-t)\,dt$$
$$-(n!(n-1)!)^{-1}\int_x^1 \int_0^1 f'(st)(x(x-s))^{n-1}(t(1-t))^n x^2\,dt\,ds.$$

Integrating now the second integral by parts with respect to s, we find (11) with (10). Obviously, $\operatorname{im} Q_n \subset \ker D^{n+1}$. The condition $P_n T_n f = 0$ for $f \in X$, which means

$$(T_n f)^{(j)}(0) = (T_n f)^{(j)}(1) = 0 \qquad (j = 0, ..., n-1)$$

is fulfilled. This completes the proof. ▌

By $Q_n = T_n - T_{n+1}D$ we obtain in particular

$$(Q_n 1)(x) = (T_n 1)(x) = ((2n)!)^{-1}(x(x-1))^n. \tag{12}$$

If $\|f\|$ denotes the Čebyšev norm of $f \in X$, then by (9), (10), and (12) it holds for all $x \in [0, 1]$ that

$$\begin{aligned} |(T_n f)(x)| &\leqslant ((2n)!)^{-1}\,\|f\|\,(x(1-x))^n, \\ |(Q_n f)(x)| &\leqslant ((2n)!)^{-1}\,\|f\|\,(x(1-x))^n \end{aligned} \tag{13}$$

and hence

$$\|T_n f\| \leqslant ((2n)!)^{-1}\,2^{-2n}\,\|f\|,$$
$$\|Q_n f\| \leqslant ((2n)!)^{-1}\,2^{-2n}\,\|f\|.$$

These inequalities are sharp by (12).

COROLLARY 3. *If $f \in C^{2N}[0, 1]$ $(N \geqslant 2)$, then the Hermite–Lagrange 2-point interpolation polynomial $P_N f$ with respect to f possesses the form*

$$(P_N f)(x) = (1-x)f(0) + xf(1) + \sum_{n=1}^{N-1} \frac{1}{n!}(a_n + b_n x)(x(1-x))^n \tag{14}$$

with

$$\begin{aligned} a_n &= \frac{(-1)^n}{(n-1)!}\int_0^1 f^{(2n)}(t)(t(1-t))^{n-1}(1-t)\,dt, \\ b_n &= \frac{(-1)^n}{(n-1)!}\int_0^1 f^{(2n)}(t)(t(1-t))^{n-1}(2t-1)\,dt. \end{aligned} \tag{15}$$

The remainder of this interpolation is given by $f - P_N f = T_N D^N f$ *with*

$$|f(x) - (P_N f)(x)| \leqslant ((2N)!)^{-1} \|f^{(2N)}\| (x(1-x))^N \qquad (x \in [0, 1]). \tag{16}$$

The proof follows immediately by (10) and (13).

4. Expansion Theorems

Theorem 4. *If* $f \in C^\infty[0, 1]$ *is a given function with the property*

$$\lim_{n \to \infty} ((2n)!)^{-1} 2^{-2n} \|f^{(2n)}\| = 0, \tag{17}$$

then the series (4) *converges uniformly in* $[0, 1]$ *to* $f(x)$, *where* a_n *and* b_n *are given by* (15). *If* $f \in C^\infty[0, 1]$ *is symmetric around* $x = \frac{1}{2}$, *that means*

$$f(x) = f(1-x) \qquad (x \in [0, 1]),$$

then for $n = 1, 2, \ldots,$

$$a_n = \frac{(-1)^n}{(n-1)!} \int_0^{1/2} f^{(2n)}(t)(t(1-t))^{n-1}\, dt, \qquad b_n = 0. \tag{18}$$

If $f \in C^\infty[0, 1]$ *is antisymmetric around* $x = \frac{1}{2}$, *that means*

$$f(x) = -f(1-x) \qquad (x \in [0, 1]),$$

then for $n = 1, 2, \ldots,$

$$a_n = \frac{(-1)^n}{(n-1)!} \int_0^{1/2} f^{(2n)}(t)(t(1-t))^{n-1}(1-2t)\, dt, \qquad b_n = -2a_n. \tag{19}$$

Proof. By (16), we obtain

$$\|f - P_N f\| \leqslant ((2N)!)^{-1} 2^{-2N} \|f^{(2N)}\|$$

for each $N \geqslant 2$. Taking the limit as $N \to \infty$, we get by the assumption (17) that

$$f = \lim_{N \to \infty} P_N f,$$

where $P_N f$ is given by (14). If f is symmetric around $x = \frac{1}{2}$, then it follows (18) by means of (15). In the antisymmetric case, (19) follows analogously by (15). This completes the proof. ∎

A function $f \in C^{\infty}[0, 1]$ is called *completely convex* [7] if, for all $n = 0, 1, ...,$

$$(-1)^n f^{(2n)}(x) \geqslant 0 \qquad (x \in [0, 1]). \tag{20}$$

A familiar completely convex function is $\sin \pi x$.

THEOREM 5. *Let $f \in C^{\infty}[0, 1]$ be completely convex. Then the series* (4) *with non-negative terms converges uniformly in* $[0, 1]$ *to* $f(x)$, *where* a_n *and* b_n, *given by* (15), *fulfill the conditions* $a_n \geqslant 0$, $a_n + b_n \geqslant 0$ $(n = 1, 2, ...)$.

Proof. If f is completely convex, then for all $n = 0, 1, ...,$

$$0 \leqslant (-1)^n f^{(2n)}(x) \leqslant c\pi^{2n} \qquad (x \in [0, 1])$$

with some constant $c \geqslant 0$ [2]. The uniform convergence of (4) is a consequence of Theorem 4, since condition (17) is fulfilled by

$$((2n)!)^{-1} 2^{2n} \|f^{(2n)}\| \leqslant ((2n)!)^{-1} (\pi/2)^{-2n} c.$$

We obtain by (15) and (20) that $a_n \geqslant 0$ and $a_n + b_n \geqslant 0$ $(n = 0, 1, ...)$. Thus all functions

$$\frac{1}{n!} (a_n + b_n x)(x(1-x))^n$$

are non-negative on $[0, 1]$. This completes the proof. ∎

Remark. For the connection between completely convex functions and Lidstone series see [1, 2].

5. SCHUR'S EXPANSION

First we consider the function $\sin \pi x$. Obviously, this function is completely convex and symmetric around $x = \frac{1}{2}$. Hence by (18), we obtain (3) and $b_n = 0$ $(n = 1, 2, ...)$. Using the estimate

$$\pi x(1-x) \leqslant \sin \pi x \leqslant 4x(1-x) \qquad (x \in [0, \tfrac{1}{2}]),$$

it follows by (3) that

$$\frac{\pi^{2n+1}}{4(2n+1)(2n-1)\cdots(n+1)} < a_n < \frac{\pi^{2n}}{(2n+1)(2n-1)\cdots(n+1)} \qquad (n = 1, 2, ...).$$

Hence the sequence a_n tends for $n \geqslant 3$ monotone decreasing to zero.

By repeated integration by parts of (3), we get the recurrence relation

$$a_{n+2} = (4n+2)\, a_{n+1} - \pi^2 a_n \qquad (n = 1, 2, ...) \tag{21}$$

with $a_1 = \pi$, $a_2 = 2\pi$ (see also [8]). Then we obtain in particular

$$a_3 = 12\pi - \pi^3, \qquad a_4 = 120\pi - 12\pi^3,$$
$$a_5 = 1680\pi - 180\pi^3 + \pi^5.$$

Using Taylor's expansion of $\sin \pi x$ at $x = 0$, (18) yields the expression (see [8])

$$a_n = \frac{1}{2} \sum_{k=n}^{\infty} (-1)^{n+k} \frac{\pi^{2k+1}}{2k(2k-1)\cdots(2k-n+1)(2k-2n+1)!}.$$

Let $r = 0, 1, ...$ be fixed and let

$$p(x) = x^r(\pi - x)^r = (\pi^2/4 - (x - \pi/2)^2)^r.$$

Observing that then

$$p^{(k)}(0) = 0 \qquad \text{for} \quad k = 0, ..., r-1,$$
$$p^{(k)}(0) = (-1)^{k-r} k! \binom{r}{k-r} \pi^{2r-k} \qquad \text{for} \quad k = r, ..., 2r,$$
$$p^{(2k-1)}(\pi/2) = 0 \qquad \text{for} \quad k = 1, ..., r$$

and using repeated integrations by parts of (3), we obtain (2):

$$\begin{aligned} a_{r+1} &= \frac{\pi}{r!} \int_0^{\pi/2} p(t) \sin t\, dt \\ &= \frac{\pi}{r!} \sum_{k=[(r+1)/2]}^{r} (-1)^k p^{(2k)}(0) \\ &= \sum_{s=0}^{[r/2]} (-1)^s \frac{(2r-2s)!}{(r-2s)!(2s)!} \pi^{2s+1}. \end{aligned}$$

We summarize:

COROLLARY 6. *Let a_n be the coefficients given by* (2), (3), *or* (21). *Then we have, for $N \geqslant 2$,*

$$0 \leqslant \sin \pi x - \sum_{n=1}^{N-1} \frac{1}{n!} a_n (x(1-x))^n \leqslant \frac{1}{(2N)!} \left(\frac{\pi}{2}\right)^{2N} \qquad (x \in [0, 1]).$$

In the case $N=6$, the resulting error of this approximation is less than 5×10^{-7}.

Now we consider other examples. The function $\sin 2\pi x$ is antisymmetric around $x=\frac{1}{2}$ and fulfills the condition (17). Hence by Theorem 4 we obtain

$$\sin 2\pi x=(1-2x)\sum_{n=1}^{\infty}\frac{1}{n!}c_n(x(1-x))^n \qquad (x\in[0,1]), \tag{22}$$

where by (19)

$$c_n=\frac{(2\pi)^{2n}}{(n-1)!}\int_0^{1/2}(t(1-t))^{n-1}(1-2t)\sin 2\pi t\,dt>0. \tag{23}$$

By repeated integration by parts of (23), we see that the coefficients c_n satisfy the recurrence relation

$$c_{n+2}=(4n+6)\,c_{n+1}-4\pi^2c_n \qquad (n=1,2,...)$$

with $c_1=2\pi$, $c_2=12\pi$ (see also [8]). By termwise integration of (22), it follows then that

$$(\sin \pi x)^2=\sum_{n=2}^{\infty}\frac{\pi}{n!}c_{n-1}(x(1-x))^n.$$

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